no-f

DADS-RS 64-0708 9

Offprint from „Metrologia"
Journal International de Métrologie scientifique — International Journal of scientific Metrology
Internationale Zeitschrift für wissenschaftliche Metrologie. Vol. 1, No. 2, April 1965, Pages 55—68
Springer-Verlag, Berlin · Heidelberg · New York

1964
1965

Standards Division, National Physical Laboratory, Teddington, Middlesex

Developments in the Accurate Measurement of High Pressures

By

R. S. Dadson, R. G. P. Greig and Angela Horner

(Received October 29, 1964) check.

With 7 Figures in the Text

Abstract

The measurement of steady high pressures in a fluid system with the highest accuracy demands the use of pressure balances (free piston gauges) of accurately known effective areas. This requires a precise knowledge of the way in which the effective areas of the piston-cylinder assemblies concerned vary due to the elastic distortion caused by the applied pressure.

Two methods which have been directed to the solution of this problem are described. The first depends on a principle of similarity as applied to the deformations of two assemblies of the same general dimensions but constructed of materials having substantially different elastic moduli. The second method makes use of measurements of the flow characteristics of the pressure transmitting fluid using two pistons having a known difference of diameter.

The distortion factors are shown to be representable as linear functions of the pressure, so that the effective area at pressure P is connected with that at zero pressure by expressions of the form

$$A_P = A_0 (1 + \lambda P)$$

where λ may be termed the distortion coefficient.

The final accuracy of the measured distortion coefficients is about $\pm 2\%$, which corresponds to an uncertainty in effective area of about ± 1 part in 10^5 at 1000 bars increasing in proportion to the pressure at higher pressures.

Some aspects of the practical calibration of pressure balances, carried out by direct balancing against assemblies calibrated by the methods described, are considered.

1. Introduction

The rapid development of high pressure techniques in the last few decades has given rise to considerably increased interest in the accurate measurement of high pressures, both in fundamental physics and chemistry and in the many associated industrial applications. In many thermodynamic studies, as for example the pressure-volume-temperature relations and virial coefficients of gases, the Joule-Thomson effect and the measurement of vapour pressures, the demands on accuracy are severe. Nevertheless until quite recently progress in high pressure measurement was much retarded compared with the measurement of the other thermodynamic variables, temperature and volume, and it is only within the last few years that some notable advance has been achieved. The object of this paper is to present an up-to-date account of some recent developments at the National Physical Laboratory which have contributed to these improvements. The discussion is restricted to the case of steady pressures.

There are two quite independent basic methods by which pressures may, in principle, be measured or established, with precision, or by which other pressure-measuring equipment may be calibrated. The first, usually represented in practice by the mercury manometer or some extension of it, determines a pressure in terms of the height of a column of liquid of known density under known conditions of gravity. In the second method, pressure is measured directly in terms of the force exerted on a surface of known area. In practice this reduces to the use of the pressure balance, or free piston gauge, in which the force due to the pressure-transmitting fluid acting on the base of a cylindrical piston, free to move in an accurately matched cylinder, is balanced by a known downard force derived from calibrated masses suitably supported on the piston. The calibration of the instrument is expressed by stating the "effective area" of the piston-cylinder assembly, and owing to the distortion caused by the applied forces this quantity may be expected to vary with pressure.

In the high pressure region proper, however, the pressure balance is virtually the only instrument in the field for practical pressure measurement of the highest accuracy, as high pressure variants of the mercury manometer are very difficult to operate even for fundamental calibration purposes. Two problems therefore present themselves:

(i) the establishment of the effective areas of suitable piston-cylinder assemblies in absolute terms at low pressures;

(ii) the determination of the changes of effective area at higher pressures due to the distortion of the assemblies resulting from the applied pressure.

With regard to (i) details are being dealt with in other publications and we shall only summarize the present position. In the more restricted field of barometric pressure the National Physical Laboratory has for many years maintained standards based on the mercury manometer and reaching an accuracy of a few microbars (Sears & Clark 1933; Elliott, Wilson, Mason & Bigg 1960). Recent work has shown that the effective areas of piston-cylinder assemblies based on comparison with a mercury manometer of a few atm range, and those calculated directly from diametral measurements on the components, are in agreement to within about 1 part in 10^5 (Dadson 1955, 1958).

The elastic distortion effect (ii) was for a long time considered to be a fundamental difficulty in the use of the pressure balance as an independent primary standard, but this situation has now been completely altered with the development of methods by which the dependence of effective area upon pressure may be determined with considerable accuracy. The present paper deals in detail with two independent methods, termed the "similarity" and "flow" methods, recently developed at the National Physical Laboratory for this purpose.

Several early attempts to measure these distortion effects by the use of high pressure mercury manometers of various forms led to very inconsistent conclusions as to the order of magnitude of the effects to be expected (HOLBORN & SCHULZE 1915; CROMMELIN & SMID 1915; KEYES & DEWEY 1927; MEYERS & JESSUP 1931; BEATTIE & EDEL 1931). MICHELS (1923, 1924, 1932) has discussed applications to the differential type of piston-cylinder assembly. The most recent, and by far the most comprehensive, investigation of this kind is that of NEWITT and his colleagues, using a 9-m pressurised differential mercury manometer installed at the Imperial College of Science and Technology (BETT, HAYES & NEWITT 1954; BETT & NEWITT 1963). The measurements, covering a range up to 700 bars*, were difficult, and the resulting distortion factors for six pressure balance assemblies of similar design varied among themselves by much more than would be expected from their construction. It seems that more extensive data will be necessary before a final assessment of the high pressure mercury column can be made. ROEBUCK & CRAM (1937) and ROEBUCK & IBSER (1954) have dealt with a recent development of the multiple-column type of mercury manometer covering the range up to about 200 bars.

The distortion errors of the "controlled-clearance" type of pressure balance used at the National Bureau of Standards, Washington, have been considered by JOHNSON & NEWHALL (1953) and by JOHNSON, CROSS, HILL & BOWMAN (1957); (see also BENNETT & VODAR 1963). It is hoped that the results of direct comparisons between the methods of calibration developed at the NBS and the NPL may be available in the near future. Accounts of the distortion errors of various designs of piston-cylinder assemblies from the point of view of elastic theory have also been published by ZHOKOVSKII (1960), SAMOILOV (1960), EBERT (1935, 1949, 1951) and TOYOSAWA (1963, 1964). These authors, however, give primary attention to the establishment of the distortion factors by calculation rather than by experiment. The present paper, on the other hand, describes direct experimental methods which are independent of other pressure standards, and practically independent of detailed elastic theory, to which appeal is made only in the calculation of small correction terms.

2. Formal Theoretical Basis

a) General

As a basis for discussion of the methods described in this paper it is useful to develop a number of formal expressions for the changes of effective area of a piston-cylinder assembly consequent on the distortion due to the applied pressure. Initially, these formulae will not involve any assumption as to the form of distortion; later, however, the results of introducing certain simplifying assumptions will be examined. Unless otherwise stated, it is assumed only that the piston and cylinder are initially straight and coaxial, that there is circular symmetry in all planes perpendicular to the axis, and that the pressure transmitting fluid in the interspace flows in accordance with the normal laws of viscosity.

The essential features of the system are shown diagrammatically in Fig. 1. The upward force due to the fluid pressure P applied to the base of the piston, corrected for the forces due to the pressure and movement of the fluid in the gap between the piston and cylinder, is balanced by the total downward force due to the load, W. We denote by r and R the radii of the undistorted piston and cylinder respectively, $u(x)$ and $U(x)$ the increases in these radii for a total applied pressure P, $p(x)$ the pressure in the interspace, and $2\,h(x)$ the radial separation, at the axial distance x measured from the lower end of the piston, l the total length of engagement, A_P the effective area of the system at the applied pressure P, and write $R-r=2\,H$, where all of H, h, u and U are very small compared with r. P and p are always to be interpreted as the amount by which the actual pressure in the system exceeds the ambient pressure — normally atmospheric — to which the balance is exposed, and the effective area as a factor of dimensions L^2 which, when multiplied into the total applied pressure, gives the total downward force provided by the load which

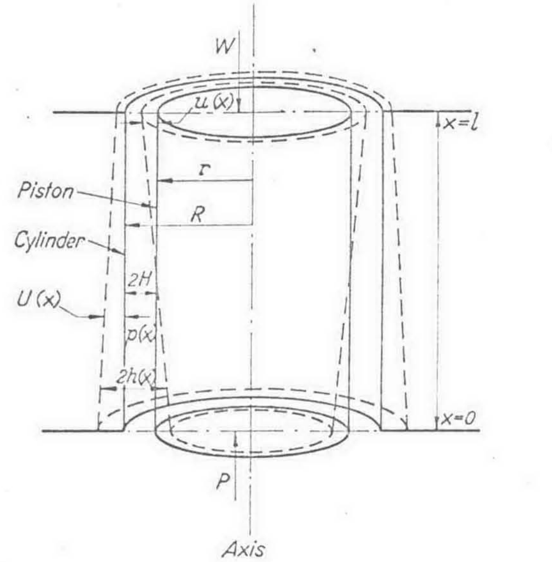

Fig. 1. Diagrammatic sketch of piston-cylinder assembly (clearance shown greatly exaggerated). —— Undistorted boundaries of piston and cylinder, - - - Distorted boundaries of piston and cylinder

is required to maintain the piston in equilibrium. For small applied pressures when distortion is negligible, we have from elementary considerations,

$$A_0 = \pi r^2\,(1 + 2\,H/r) \tag{2.1}$$

neglecting second and higher-order terms in $2\,H/r$, where A_0 is the effective area at zero pressure.

To obtain the more general formulae when distortion is present we note that the fluid forces acting on the piston have the following components:

a) upward force due to applied pressure on base of piston

$$P\pi r^2\,[1 + 2\,u(0)/r]\ ;$$

b) upward force due to fluid friction on flanks of piston

$$2\,\pi r \int_0^l \left(-h\,\frac{dp}{dx}\right) dx\ ,$$

$$= 2\,\pi r \int_0^l \left[-\frac{d(ph)}{dx} + \frac{p}{2}\left(\frac{dU}{dx} - \frac{du}{dx}\right)\right] dx\ ;$$

c) upward force due to vertical component of applied pressure on flanks of piston

$$2\,\pi r \int_0^l p\,\frac{du}{dx} \cdot dx\ .$$

Thus the total upward force acting on the piston is

$$P\pi r^2\,[1 + 2\,u(0)/r] + 2\,\pi r \int_0^l \left[-\frac{d(ph)}{dx} + \right.$$
$$\left. + \frac{p}{2}\left(\frac{dU}{dx} + \frac{du}{dx}\right)\right] dx\ ,$$

---

* 1 bar = 10^6 dyn/cm^2 = 10^5N/m^2.

or

$$P\pi r^2\left[(1+2H/r)+\frac{U(0)+u(0)}{r}+\frac{1}{rP}\int_0^l p\left(\frac{dU}{dx}+\frac{du}{dx}\right)dx\right].$$

The formula for the effective area is now

$$A_P=\pi r^2\left[(1+2H/r)+\frac{U(0)+u(0)}{r}+\right.$$
$$\left.+\frac{1}{rP}\int_0^l p\left(\frac{dU}{dx}+\frac{du}{dx}\right)dx\right] \tag{2.2}$$

of which expression (2.1) is a special case with U and u zero. We may evidently obtain (2.2) more directly by visualising the neutral surface as the effective boundary of the piston in which case the frictional force corresponding to b) above vanishes and we are left simply with the pressure forces acting on the effective boundaries of the piston. We also have the equivalent form

$$A_P=\pi r^2\left[1+2\,u(0)/r+\right.$$
$$\left.+\frac{2}{rP}\left(\int_0^l -h\,\frac{dp}{dx}\cdot dx+\int_0^l p\,\frac{du}{dx}\cdot dx\right)\right] \tag{2.3}$$

which is convenient for use when the integral $\int_0^l -h\,\frac{dp}{dx}\,dx\left(or\int_0^P h\,dp\right)$ is of interest, as is the case, for example, when the flow method (section 5) is considered.

b) *The effects of special assumptions*

The problem of calculating the actual changes of effective area of practical designs of piston-cylinder assembly, on the basis of the above general formulae, is complicated. It would be necessary to know the interrelated quantities u, U and p as functions of x, and since the pressure gradient dp/dx is governed by the normal equation of viscous flow (see equation 5.1), the pressure dependence of the coefficient of viscosity would also need to be taken into account. It is not, however, the aim of the present paper to attempt such calculations, but rather to describe direct experimental methods for the accurate determination of the distortion factors with the minimum of assumptions regarding the detailed behaviour of the system. We therefore consider only certain special cases which are useful in the applications which follow.

A useful approximation may be derived from the foregoing equations by assuming that the component of $u(x)$ or $U(x)$ due to the fluid pressure in the interspace between piston and cylinder may be taken to be proportional to the pressure $p(x)$ *at the same position*. The relevant terms in the integrals on the right hand side then become integrable without the necessity for any further knowledge of the actual functional forms of $u(x)$, $U(x)$ or $p(x)$. There is fair support from elastic theory for this assumption, more especially in the case of the solid cylinder in which the length is large compared with its radius, a condition which applies to the pistons of most pressure balance assemblies other than those catering for only a low range of pressure. CHREE (1889, 1901) has given polynomial solutions for the equilibrium of a finite solid cylinder for cases in which the lateral pressure is either a linear or quadratic function of the axial co-ordinate. The conditions are satisfied by functions $u(x)$ and $p(x)$ which are accurately proportional, provided the normal tractions over the flat ends, instead of being identically zero, are assumed only to average to zero. By Saint-Venant's principle, however, the effect of this disturbance will be appreciable for only a short distance from each end, and may be neglected if the ratio of length to radius is considerable. The constant of proportionality is the same as in the case of uniform pressure on a solid cylinder of infinite length. FILON (1902) has obtained solutions for pressure distributions expressed in series of trigonometric functions of x which lead to a similar result provided the wavelengths involved are fairly large compared with the radius. The effects of discontinuous pressure distributions, or narrow bands of applied pressure, have also been discussed (BARTON 1941; RANKIN 1944; TRANTER & CRAGGS 1947), with the general result that even the effects of discontinuities are largely lost at an axial distance of only about half the radius. If, therefore, the pressure changes along the length of the assembly are reasonably smooth, no great error is likely to be incurred by applying this assumption to the *piston* of the assembly. Taking into account the additional change of radius due to the end thrust on the piston, it is easily shown that the relevant terms involving u on the right hand side of equation (2.2) reduce to $P\,(3\,\sigma-1)/2\,E$ where E and σ are respectively Young's modulus and Poisson's ratio, so that we now have, using also (2.1),

$$A_P=A_0\left[1+\frac{P(3\,\sigma-1)}{2\,E}+\frac{U(0)}{r}+\frac{1}{rP}\int_0^l p\,\frac{dU}{dx}\cdot dx\right]. \tag{2.4}$$

Another useful form, obtained directly from (2.3), is

$$A_P=\pi r^2\left[1+\frac{P(3\,\sigma-1)}{E}+\frac{2}{rP}\int_0^P h\,dp\right]. \tag{2.5}$$

Note at $h \to 0$ as in Controlled gauge this exp $\to$ that of Johnson et al

The application of a similar assumption to deal with the effects of internal pressure in a hollow cylinder with thick walls is less secure. CHREE (1901) has given a corresponding solution with $U(x)$ and $p(x)$ proportional for the case where $p(x)$ is a linear function of x, but its validity would depend on the conditions assumed at the ends. The case of a discontinuous distribution of pressure has been considered briefly by TRANTER (1946). In the ideal case of a cylinder whose length is large compared with its radius and wall thickness, where the working section is removed some distance from the points of attachment of the ends, and the pressure distribution is reasonably smooth, a useful approximation may result. Proceeding from equation (2.4), and taking for definiteness the case where the cylinder walls are not subjected to longitudinal stress, we then obtain (LOVE 1952), denoting by R' the outer radius of the cylinder,

$$A_P=A_0\left\{1+\underbrace{\frac{P}{2\,E}\,(3\,\sigma-1)}_{\text{(piston)}}+\underbrace{\frac{P}{2\,E}\left[\frac{(1+\sigma)\,R'^2+(1-\sigma)\,R^2}{R'^2-R^2}\right]}_{\text{(cylinder)}}\right\}$$

or, combining the distortion terms,

$$A_P = A_0\left[1 + \frac{P}{E}\left(2\,\sigma + \frac{R^2}{R'^2 - R^2}\right)\right] . \qquad (2.6)$$

In the limiting case with R'/R effectively infinite this reduces to the simple expression

$$A_P = A_0\left(1 + \frac{2\,\sigma}{E}\cdot P\right) . \qquad (2.7)$$

Equations (2.4) to (2.7) are a useful basis for the development of certain small correction terms which arise in the theory of the similarity and flow methods.

3. The Similarity Method

a) Principle of the method

In normal practice the assemblies for which calibrations are principally required are constructed of steel. The principle adopted in the similarity method is first to determine the ratio of the effective area of the steel piston-cylinder assembly of given type, at a series of pressures, to that of a precisely similar assembly constructed of a material having a substantially different elastic modulus. This procedure determines the difference between the distortion factors of the two assemblies as a function of pressure. A second relation — the quotient of the two distortion factors — is obtained from measurements of the elastic moduli of the two materials. The combination of these results then allows the distortion factor of each assembly to be derived, as a function of pressure, in absolute terms.

b) Ideal theory of the similarity method

In its ideal form the similarity method is extremely simple, and involves no assumption regarding the form of distortion of the assembly when under pressure. In the ideal situation the two materials are regarded as elastically isotropic, with linear stress-strain relationships and identical Poisson's ratios over the range of stress involved. The two assemblies are assumed to be constructed to the same principal dimensions and to have accurately straight and circular pistons and cylinder bores. Ideally, the initial radial separations between the components of the two assemblies should be in inverse ratio to their elastic moduli, although it is found in practice that this condition is not critical. These conditions ensure that, as the distortion changes with increasing pressure, the annular channels between piston and cylinder will remain similar in form and that consequently the pressure distributions along the lengths of the channels will always remain the same for the same total applied pressure.

If these assumptions are realised the distortion terms in the expressions for the effective areas will remain in a fixed numerical ratio as the pressure is varied. In other words the effective areas A_P and B_P of the two assemblies at the applied pressure P may be written in the form,

$$A_P = A_0[1 + \lambda_A f(P)]\,;\; B_P = B_0[1 + \lambda_B f(P)] \qquad (3.1)$$

where λ_A, λ_B are constants in inverse ratio to the elastic moduli, and $f\,(P)$ is a function of the applied pressure of which the form is unknown but is the same in both cases. Bearing in mind that the distortion terms are normally very small compared with unity, the ratio of the areas may be expressed in the form

$$\frac{A_P}{B_P} = \frac{A_0}{B_0}\,[1 + (\lambda_A - \lambda_B)\,f(P)] \qquad (3.2)$$

and writing $\lambda_B = k\lambda_A$, where k is a constant, we obtain

$$\frac{A_P}{B_P} = \frac{A_0}{B_0}\,[1 + (1 - k)\,\lambda_A f(\mathrm{P})] . \qquad (3.3)$$

The ratio A_P/B_P, and consequently the function $(1 - k)\;\lambda_A\,f\,(P)$, may be determined easily and with high precision by simply measuring the loads on the two pistons when the assemblies are balanced against one another and in equilibrium at the same pressure, and carrying out this procedure at a series of pressures over the appropriate range. The quotient, k, of the elastic moduli may be determined by the standard methods for the measurement of elastic constants. It is clear that in the ideal conditions postulated these two procedures suffice to establish the values of the distortion terms $\lambda_A\,f(P)$ and $\lambda_B\,f(P)$ to an accuracy limited only by the sensitivity of the balancing process and the precision to which the elastic constants are known. In general it is found to be the second factor which eventually limits the accuracy attainable, and to obtain the best precision k should evidently differ substantially from unity.

It is of particular interest that the rheological properties of the pressure transmitting fluid — e. g. dependence of coefficient of viscosity upon pressure — are entirely eliminated in the similarity procedure.

In order to simplify further discussion it is useful at this point to anticipate one practical result of the investigation, viz. that in most cases the distortion is very closely represented by a linear function of the applied pressure so that we may normally replace $f(P)$ by P, when the quantities λ_A and λ_B may be regarded simply as pressure coefficients having the dimensions $(\mathrm{pressure})^{-1}$. Thus we may write instead of (3.1), $A_P = A_0\;(1 + \lambda_A P)$ etc., in all but exceptional cases.

c) Effect of departures from the ideal conditions

It would be a somewhat fortunate coincidence if the ideal assumptions were completely realised in a pair of actual metals having a sufficiently large ratio of elastic moduli, and also adequate tensile strengths, to justify their use in practice, and it is necessary to consider to what extent minor departures may be tolerated, or whether reliable correction terms can be developed. Materials showing appreciable elastic anisotropy are hardly worth consideration owing to the greatly increased complexity of the distortion of the system, and the labour of determining the complete set of elastic constants over a wide range of stress. Again, a pronounced departure from a linear stress-strain relation would introduce awkward complications; small departures may be tolerable, subject to a corresponding uncertainty in the value of the elastic modulus. In the case of a moderate difference in the values of the Poisson's ratios, however, it is not difficult to formulate a correction term. This is small and need only be evaluated approximately. For this purpose we make use of the formula (2.4), and express the distortion coefficients in the form $\lambda_A = \theta_A + \varphi_A \cdots$ where

$$\theta_A = (3\,\sigma_{(A)} - 1)/2\,E_{(A)}\;\ldots \qquad (3.4)$$

and φ_A is that part of λ_A which is explicitly dependent on the deformation of the cylinder. As the ratio R'/R is fairly large in the actual cases considered, the main term expressing the cylinder distortion is proportional to $(1 + \sigma)/E$, i. e. to $1/G$ where G is the modulus of rigidity.

Interpreting k as the quotient of the two moduli of rigidity, we now have

$$\begin{aligned}\lambda_A - \lambda_B &= \varphi_A - \varphi_B + \theta_A - \theta_B \\ &= (1 - \mathrm{k})\,\varphi_A + \theta_A - \theta_B \\ &= (1 - \mathrm{k})\,\lambda_A + (k\theta_A - \theta_B) \qquad (3.5)\end{aligned}$$

determining λ_A in terms of the difference coefficient $\lambda_A - \lambda_B$ established by the balancing procedure, the value of k, and the correction term $(k\theta_A - \theta_B)$.

d) Extension to the use of three materials

In the first series of experiments the material adopted for the comparison assemblies was a form of aluminium bronze, known commercially as "hydurax", the modulus of rigidity of which was lower than that of steel in the ratio 1:1.44. The Poisson's ratio was rather higher than that of steel (see Tab. 1 for further details). It was apparent that a check involving a third material, differing substantially in elastic properties from those used hitherto, would provide a valuable test of the accuracy of the similarity method. An even better check would naturally be provided by a completely independent pair of materials. This latter extension has not so far been found practicable as the choice of materials possessing all the qualities required is limited. It has been found possible, however, to extend the procedure to include three materials, the third being an alloy of tungsten known commercially as "GEC Heavy Metal". This material proved to have a high degree of isotropy and a Poisson's ratio very close to that of the material used for the steel assemblies. The elastic moduli exceed those of steel in about the ratio 1.75:1 and it was of advantage that in this case the comparison should involve a material having a modulus higher than that of steel in contrast to the former comparisons in which the reverse held.

In discussing this extension of the method it will be convenient to refer to the steel, bronze and tungsten assemblies by the initial letters S, B and T respectively. With a group of three materials, the distortion coefficient of any one assembly, say S, may be reached by three different routes, two of them direct — i. e. involving direct comparisons with the other two assemblies B and T — and the other indirect. In the latter procedure the distortion coefficient of one of the other two assemblies, say T, is first determined by applying the similarity principle to B and T, and the coefficient for S is then obtained by simple addition of the difference coefficient for S and T. It is of interest to note that the indirect procedure leads to the distortion coefficient of the assembly chosen, i. e. S, without any appeal to the elastic constants of the material of S. These three derivations are not entirely independent but, since the six independent elastic moduli involve five independent ratios, no one result is in general deducible from the other two. Proceeding on the lines of equation (3.5) and denoting by $\lambda_S \ldots$ the true values of the distortion coefficients, $G_S \ldots$ the moduli of rigidity, $\theta_S \ldots$ the corresponding correction terms given by equation (3.4), $\lambda_{SB}\,(= \lambda_S - \lambda_B) \ldots$ the difference coefficients determined by the balancing experiments, and $k_{SB} = G_B/G_S \ldots$, we have for the three possible experimental values, λ'_S, λ''_S and λ'''_S, of λ_S, the equations

$$\lambda'_S\,(k_{BS} - 1) = \lambda_{BS} - \theta_B + k_{BS}\,\theta_S \qquad (3.6)$$

$$\lambda''_S\,(k_{TS} - 1) = \lambda_{TS} - \theta_T + k_{TS}\,\theta_S \qquad (3.7)$$

for the direct comparisons, and for the indirect

Table 1. *Summary of elastic constants*

Material	Modulus of rigidity (G) (dyn/cm²)*		Young's modulus (E) (dyn/cm²)*		Poisson's ratio (σ) (Ultrasonic pulse method)
	Torsion extensometer method	Ultrasonic pulse method	Extensometer method (mean of results for tension and compression)	Ultrasonic pulse method	
Steel (K 9) (hardened and tempered)	7.86×10^{11}	7.92×10^{11}	20.6×10^{11}	20.5×10^{11}	0.295
Aluminium bronze ("hydurax")	5.45×10^{11}	5.38×10^{11}	$14.4_5 \times 10^{11}$	$14.3_3 \times 10^{11}$	0.333
Tungsten alloy ("GEC heavy metal" – specific gravity 18)	$13.5_5 \times 10^{11}$	$14.2_5 \times 10^{11}$	36.1×10^{11}	36.7×10^{11}	0.286_5

* 1 dyn/cm² = 0.1 N/m².

$$\lambda'''_S\,(k_{BT} - 1) = \lambda_{BT} - \theta_B + k_{BT}\,\theta_T + \lambda_{ST}\,(k_{BT} - 1). \qquad (3.8)$$

Transposing these equations and making use of the subsidiary relations

$$\left.\begin{aligned} k_{BS}\,k_{SB} = 1 \ldots\,; \quad k_{BS}\,k_{ST}\,k_{TB} = 1\,; \\ \lambda_{BS} = -\,\lambda_{SB} \ldots\,; \quad \lambda_{BS} + \lambda_{ST} + \lambda_{TB} = 0\,; \end{aligned}\right\} \qquad (3.9)$$

we eventually obtain

$$(\lambda'_S - \lambda'''_S)\,(1 - k_{SB}) = (\lambda'''_S - \lambda''_S)\,(k_{ST} - 1). \qquad (3.10)$$

Since $(1 - k_{SB})$ and $(k_{ST} - 1)$ are both positive it easily follows from this equation that the three values λ'_S, λ''_S and λ'''_S must either be all equal or all unequal, and that the indirect value λ'''_S must be intermediate between the two direct values, whatever the nature of the experimental errors*. The practical significance of various possible errors is examined in more detail in section 4 b).

e) Determination of elastic constants

The elastic constants utilised in the investigation were measured in the Strength of Materials Section of the Basic

* We ignore cases where either k_{SB} or k_{ST} is so close to unity that experimental errors might cause a change of sign of $(1\text{-}k_{SB})$ or $(k_{ST}\text{-}1)$ since such conditions would not be acceptable as a basis for the similarity method.

Physics Division of the National Physical Laboratory, and included results obtained by the ultrasonic pulse method (Markham 1957) as well as by the standard static methods giving the stress-strain relations over a wide range of stress. Young's modulus was measured both in tension and compression using a Martens type rhomb and mirror extensometer. The modulus of rigidity was determined by means of an NPL design of torsion extensometer in which readings were taken either with an autocollimator or with the normal arrangement of scale and telescopes.

Precautions were taken to ensure that the samples used for the preparation of test pieces were sufficiently representative of the material used in the piston-cylinder assemblies. Wherever possible they were selected from the same piece or batch of material. In cases where this was impracticable, material of similar composition was used, care being taken that any heat treatments involved were adequately reproduced. A study by Brown, Cole & Markham (1957) on the effects of heat treatment and tempering on the elastic moduli of the steels concerned illustrates the significance of these effects.

The results of the elastic modulus measurements are summarised in Tab. 1. On the whole the agreement between the ultrasonic and static methods is good, the discrepancies rarely exceeding 1 or 2%. It seemed desirable, however, to decide on a consistent basis for the choice of the actual values to be adopted in practice, especially as regards the values of G and σ which are particularly important in the applications to the similarity method. It was decided, after consultation with experts in the field of elastic properties, to proceed as follows:

i) For the modulus of rigidity, to adopt the static values taken over a wide range of stress, as being those most likely to be representative of the conditions obtaining in practice when the system is subjected to sustained forces. It is pertinent to note that as we are interested only in the *ratio* of the values of G for a pair of materials, certain types of systematic error in the elastic measurements will be eliminated.

ii) For Poisson's ratio, to adopt the values obtained by the ultrasonic method in which this quantity is given directly in terms of the observed wave velocities. This value is likely to be considerably more accurate than one derived indirectly from static measurements of E and G since, as these are determined by different experimental procedures, their ratio may be subject to a systematic error. Since $E/G = 2\,(1 + \sigma)$ and σ is normally intermediate between 1/3 and 1/4, any error in E/G would entail an error proportionately 4 or 5 times larger in σ. It may be noted, however, that even if the actual value of E/G were somewhat in error the relation between the loads and displacements would still help to show up any important *variation* in σ over the range of stress, so that the static results provide useful evidence on this point.

The ultrasonic measurements provide direct information on the elatic isotropy of the material. This was found to be satisfactory in the case of all three materials considered in this investigation.

The relations between displacement and applied force given by the extensometer measurements showed a satisfactory degree of linearity, and freedom from important hysteresis effects, with the exception of the tungsten alloy at high stresses. When tested under the condition of a rising series of values of stress, this material exhibited departures from linearity, principally for stresses above about 1600 bars (1.6×10^8 N/m^2), which seemed consistent with some degree of plastic deformation. Series taken in descending order of stress, however, showed a much closer approximation to linear behaviour, indicating a modulus reasonably consistent with that obtaining over the lower range of stress, i. e. before the appearance of the anomalous permanent set. This point is further discussed in the next section, where a variation of the balancing procedure used in the similarity method, to take account of this anomaly, is described.

f) *Experimental method*

As previously remarked, the effective areas of the piston-cylinder assemblies of the two different materials have been compared by direct balancing on a common pressure system as this is the most convenient method assuming that two complete pressure balances are available*.

For the purposes of the present work the equilibrium state of a piston-cylinder assembly is defined to be that in which the piston is falling at such a rate as exactly to compensate for the volume of fluid lost by the natural leakage through the inter-space between the piston and cylinder. In the case of two assemblies balanced against one another, these conditions imply that there is no movement of fluid through the connecting line. Leaks in other parts of the system must of course be carefully controlled if these equilibrium conditions are to be reproduced unambiguously. The accuracy of the balancing process is normally of the order of a few parts in 10^6.

The dependence of the effective area on temperature has been found to be adequately represented by the area coefficient of thermal dilatation which, in the case of steel assemblies, amounts to a change of about 2.3 parts in 10^5/ °C. The temperatures of the piston-cylinder assemblies were measured to within about 0.05 °C.

even due $\simeq \frac{1}{10^6}$ to temp

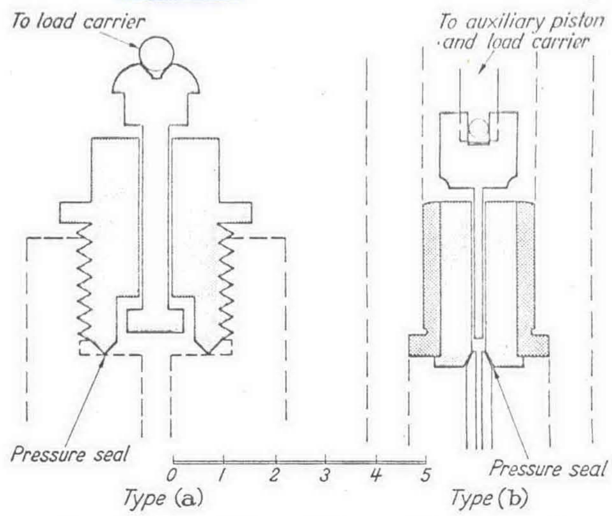

Fig. 2. Diagrams of piston-cylinder assemblies (Scale of cm)

Some obvious small corrections to the loads on the two assemblies may be necessary to account for:

i) any difference of level of the two pistons;

ii) buoyancy effects due to any submerged portions of the piston of other than the working diameter;

iii) surface tension at the meniscus at the upper end of the piston.

Since the comparison is between assemblies of the same nominal dimensions, the corrections involved in ii) and iii) will normally cancel out, or nearly so.

Two rather different types of piston-cylinder assembly have been used in the present work, and these are shown diagrammatically in Fig. 2, a) and b). Units of type a) have been used over the range of pressure up to about 3000 bars, the assemblies having nominal effective areas of 0.05, 0.02 and 0.01 in^2* and differing only in the diameter of the piston and cylinder bore. The units of type b), which have been used mainly for the higher part of the pressure range – i. e. from about 1500 to 6000 bars – were of nominal area 0.005 in^2*.

The piston-cylinder units of type a) are attached to the support column by screwing into a collar shown in outline in Fig. 2, the pressure seal being effected between an annular projection at the base of the assembly and a flat shelf at the upper end of the column. In order to avoid any possibility of anomalous effects due to a discontinuity in the elastic modulus at the junction, the support column used in association with any particular assembly was constructed of the same material as the assembly itself. In the units of type b) the housing, also shown in Fig. 2, was rather different. The main cylinder block

* It should be noted that the balancing process is not in itself fundamental to the similarity procedure. The essential condition is that the equilibrating loads on the two assemblies should be determined for *exactly* the same pressure. It would be possible, though more difficult, to do this by determining the load on each assembly separately when exposed to an accurately reproducible pressure identified, for example, by a phase transition of a pure substance. If two complete balances were not available it might well be necessary to resort to some such method.

* The approximate metric equivalents are:
0.05 in^2 = 0.322 cm^2; 0.02 in^2 = 0.129 cm^2;
0.01 in^2 = 0.0645 cm^2; 0,005 in^2 = 0.0322 cm^2.

was held with a tight press fit in an outer jacket and, to investigate the effect of the latter, tests were made both with the outer jacket of the comparison assembly of the same material (tungsten) as the inner block, and also with a jacket of high tensile steel. These two arrangements showed no appreciable difference as regards the distortion factor.

All diametral measurements required on the pistons and cylinders were carried out in the Engineering Metrology Section of the Standards Division of the National Physical Laboratory by direct comparison with high quality slip gauges, the sizes of which are known to about $\pm 10^{-6}$ in ($\pm 0.025\ \mu$m) (NPL Ann. Rep. 1919: TAYLERSON 1955).

The main part of the load on the piston was applied in the familiar manner by annular masses stacked on a cylindrical carrier of the overhang type supported on the upper end of the piston by a steel ball. In order to minimise friction the assemblies were always operated with the piston and load system in free rotation. The speed of rotation is not in general critical for assemblies of the types used in the present measurements but for definiteness a speed in the range 30 – 40 rev/min was normally adopted. Piston-cylinder assemblies occasionally exhibit anomalous effects due to small helical errors on the piston surface – often referred to as "corkscrewing" – which have the effect of adding a spurious component – positive or negative according to the direction of rotation – to the load. These effects are easily identified and in order to eliminate them measurements were always made using both directions of rotation, and the mean ~~value adopted~~. ~~Any assembly~~ showing a considerable degree of asymmetry of this kind would have been rejected as unsuitable for measurements of the accuracy and reproducibility necessary for the present work.

In carrying out the balancing experiments the fall of the pistons was observed either by the use of optical magnification, or electronically using a capacitance method.

The normal practice in taking observations over any given range of pressure was first to take a series in rising order of pressure and to follow this as soon as possible by a repeat in descending order. In general these series showed no systematic divergence and hysteresis effects were negligible. There was, however, one exception to this rule, applying to comparisons involving the tungsten base material at pressures above about 3000 bars. In this case the rising series of points over the upper part of the pressure range showed a tendency to curve away from the initial straight line in the sense of an abnormally large increase in area on the part of the tungsten assembly. This abnormal component of the deformation recovered only very slowly on removal of the pressure, and it was found that if, after exposure to the maximum pressure, a relatively rapid series of readings was taken in descending order, these approximated well to a straight line which, moreover, was sensibly parallel to the initial portion where hysteresis was not appreciable. As already pointed out, the elastic constant measurements on the tungsten base alloy showed very similar characteristics, with anelastic effects over the higher ranges of stress but providing reasonably consistent values of the elastic modulus from the series of readings taken with diminishing stress. It was considered justifiable, therefore, to regard the descending series as being fairly representative of the elastic behaviour of these assemblies, in so far as this enters into the similarity procedure. On this basis measurements with the steel and tungsten assemblies were extended up to the region of 6000 bars. The practicability of using some more recently developed alloys of high modulus is being considered for possible further extensions of the method.

4. Results of the Similarity Method

a) Measurements involving two materials for the range up to 3000 bars

Some account of the earlier measurements in this series has been given in two former papers (DADSON 1955, 1958) but for completeness the main features are summarised below.

Fig. 3 illustrates the results obtained with a series of piston-cylinder assemblies of type a) – Fig. 2 – covering three different ranges of pressure. The changes in effective area are shown as parts in 10^5 of the area at zero pressure, and in two cases results are given for different transmitting fluids.

As was mentioned earlier the distortion factors for assemblies of this type may be very closely represented as linear functions of the applied pressure, the dispersion of the experimental points rarely amounting to more than ± 1 part in 10^5.

It will also be apparent that for a given fluid the distortion coefficients for assemblies having different cylinder bores are very similar, the coefficient λ_S being normally in the region 4×10^{-7}/bar. The normal manufacturing tolerances on this type of assembly seem to involve little variation in the distortion coefficient, the values for a substantial group for the same transmitting fluid having been found to vary by only a few percent.

A point of interest arises in connection with the use of different fluids, when, as illustrated in Fig. 3,

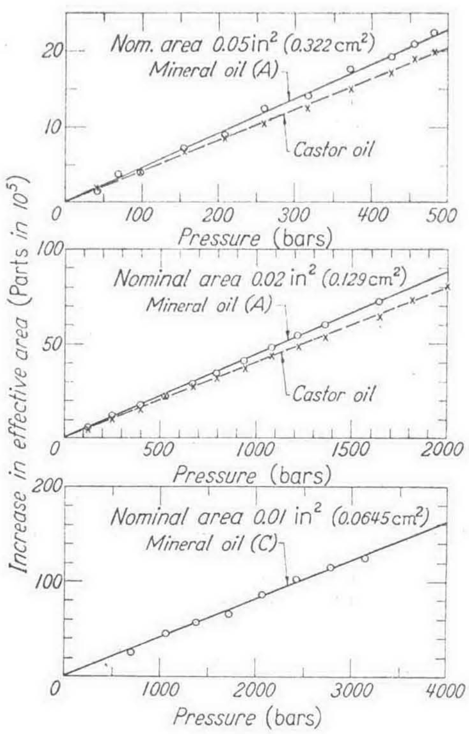

Fig. 3.
Distortion factors of a group of steel piston-cylinder assemblies of type a

some variation of the distortion coefficient may occur. It would seem that these effects must be connected with differences in the functional form of the dependence of the coefficient of viscosity upon pressure and its resulting influence on the pressure distribution in the interspace between piston and cylinder. In the discussion of the formal theory of the pressure balance earlier in this paper the effect was examined of assuming that the components of the radial displacements of the surfaces of the piston and cylinder at a given position due to the fluid pressure in the interspace could be taken as proportional to the pressure at the same position. Reasons were adduced that this assumption was unlikely to be much in error in the case of the piston, but was less secure in the case of the cylinder. It is an immediate consequence of this assumption – see equation (2.6) – that the distortion factor is independent of the actual pressure distribution in the interspace, and should therefore be independent of the transmitting fluid. The experimental results thus provide evidence that the assumption in question is not entirely correct, at least for the assemblies of type a), and it will be the cylinder, the lateral dimensions of which are not small compared with the length of the working section of the bore, where the principal limitation will arise. If the cylinders were appreciably longer compared with their wall thickness, and the region of attachment were located further away from the working portion, the dependence on the nature of the fluid might well be reduced. Although the changes so far observed are not very large, they are sufficient to require that any standard calibration of a piston-cylinder assembly intended for work of high accuracy must be associated with the particular fluid used. This is an aspect of the pressure balance on which more data would be useful.

b) Results of measurements involving three materials with discussion of errors

The three-material procedure has been carried out for two pressure ranges — 500 and 1200 bars — employing assemblies of type a) — Fig. 2 — of nominal areas 0.05 and 0.02 in² respectively, using castor oil as the pressure transmitting fluid. The results of these measurements are summarised in Tab. 2 in which are shown the values of the distortion coefficients for the steel assemblies determined both by the direct and indirect methods. Over the pressure range in question the dependence of disrortion on pressure was closely linear, with no appreriable hysteresis effects. The actual coefficients given are best fits by least squares to some four to six sets of data. It is worthy of note that it has been verified by direct balancing that the distortion coefficients of the two steel assemblies concerned are actually equal to within 1%. The total dispersion of the results is in the region ± 4%, but it will be seen that there is evidence that the direct comparisons involving bronze (λ'_S) are subject to more scatter than the remainder. This result is not surprising since, from the point of view of the influence of possible uncertainties in the elastic constants, this comparison is in every way at a disadvantage relative to the other two. Since the factor k has here its smallest value (= 1.44), and the comparison is with an assembly having a *larger* distortion, the operative factor in equation (3.6), viz $(k - 1)$, is particularly sensitive to an error in k. The fact that the Poisson's ratios are somewhat different is also not an advantage, although the correction factor already discussed should take account of this. Making use of equations (3.6) to (3.8) and introducing the actual numerical values of k_{BS} ..., it is easily shown that an error of x% in the relevant ratio of elastic moduli (k) leads to percentage errors in the three values, λ'_S, λ''_S and λ'''_S of the distortion factor, of about 3.3x, 1.4x and 0.9x respectively. In this respect therefore, the direct comparison using S and T and the indirect comparison, would be expected to show an appreciable advantage over the direct comparison using S and B. Considering now the errors associated with the correction terms θ_S ... of equations (3.6) to (3.8), introduced to allow for differences of Poisson's ratio, some advantage may lie with the direct comparison using S and T in which the two Poisson's ratios are nearly equal, the correction term in this case amounting to only about 2% of the total distortion factor.

Table 2. *Results of three-material experiments*

Nominal effective area	Pressure range (bars)	Distortion coefficient of steel assembly for castor oil (bar⁻¹): Direct comparison with bronze (λ'_S)	Direct comparison with tungsten (λ''_S)	Indirect comparison (λ'''_S)
0.05 in² (0.322 cm² approx.)	500	$4.2_1 \times 10^{-7}$	$4.0_0 \times 10^{-7}$	$4.0_9 \times 10^{-7}$
0.02 in² (0.129 cm² approx.)	1200	$3.9_6 \times 10^{-7}$	$4.1_0 \times 10^{7}$	$4.0_5 \times 10^{-7}$
Mean results for above cases		$4.0_8 \times 10^{-7}$	$4.0_5 \times 10^{-7}$	$4.0_7 \times 10^{-7}$

The data of Tab. 2 are therefore seen to be consistent with the assumptions that the main errors involved are associated with the values adopted for the elastic moduli, and that the ratios of these are known to the order of ± 1 or 2%, the corresponding distortion coefficients being contained within a dispersion of about ± 4%. If, however, the two most favourable comparisons (λ''_S and λ'''_S) are selected, and the mean taken, the final result is unlikely to be in error by more than about 2%. In the practical application of the results this procedure has been adopted.

c) Extension to pressures of 6000 bars

The extension of the similarity method from 3000 to the region of 6000 bars has been carried out entirely with assemblies of type b), of nominal area 0.005 in², those of type a) being normally restricted to use below 3000 bars. The experimental value of the distortion coefficient is $3.0_2 \times 10^{-7}$/bar, and is thus appreciably smaller than the figure for assemblies of type a) averaging at about $4.0_6 \times 10^{-7}$/bar.

The form of the type b) assemblies approximates more closely to the "ideal" piston-cylinder combination. In considering the formal theory in Section 2 it was noted that a very simple approximation to the distortion factor could be derived on the assumption that the radial displacements of the piston and cylinder surfaces at any position due to the fluid pressure in the interspace are proportional to the pressure at that position, and the limitations of this assumption were discussed. Inserting the appropriate numerical values in equation (2.6) the distortion coefficient so deduced, assuming a ratio of external to internal cylinder diameter of 10:1, is about 2.9×10^{-7}/bar. The close approach of this figure to the experimental value for the type b) assemblies certainly suggests that the assumptions involved in the "naive" theory are not greatly in error in this case. There are, however, some features of the actual cylinder, notably the stresses on the end surfaces, which are not taken into account in the calculation so that the rather close agreement observed in this particular instance may be partly fortuitous.

The application of the similarity method determines the distortion coefficients of the type a) and type b) assemblies quite independently of one another. Since the two types are found to have appreciably different coefficients, a direct comparison, e. g. by balancing a steel assembly of type a) against one of type b), is now able to provide an additional check of the overall accuracy of the procedure. The results of experiments on these lines are shown in Fig. 4, which compares the values of the distortion factors of a type b) assembly derived in two independent ways:

i) by direct application of the similarity method to the type b) assembly, and

ii) by comparison of the same type b) assembly with assemblies of type a), the distortion factors of which had previously been determined by direct application of the similarity method.

It will be seen that the results obtained by the two methods are practically indistinguishable; the actual mean values of several determinations of the distor-

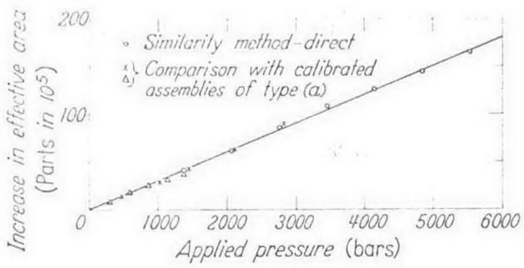

Fig. 4. Distortion factor of assembly of type b determined by two methods

tion factor for the type b) assembly are $3.0_2 \times 10^{-7}$ and $2.9_5 \times 10^{-7}$/bar for procedures i) and ii) respectively. This independent check thus supports the estimates of accuracy put forward in the foregoing section.

d) *Practical applications*

Once the effective area of a pressure balance assembly has been measured in absolute terms as a function of pressure over a given range, it is possible to calibrate almost any other assembly covering the same range, and using the same pressure transmitting fluid, by the process of direct balancing. In the course of the present investigation a large number of individual balances of different patterns have been calibrated, including many for other users. Balances involving piston-cylinder assemblies of types a) and b) — Fig. 2 — have already been discussed. These show, for a given fluid, fairly consistent distortion coefficients, typified by the values given above in sections 4 b) and 4 c). In such cases, it may be sufficient for many purposes to take an average figure as typical of assemblies of a given pattern.

Another type of balance in common use, of which a considerable number have been calibrated, is that employing a simple piston-cylinder assembly consisting of a bronze cylinder combined with a steel piston. This type also exhibits fair consistency as regards dependence of effective area upon pressure, the distortion coefficient being about 8×10^{-7}/bar.

Calibrations have also been made of a number of differential piston-cylinder assemblies of the well known form shown diagrammatically in Fig. 5. In this type of assembly the actual effective area is the difference between the effective areas of the two constituent piston-cylinder combinations, the upper combination being varied in diameter to suit the desired pressure range. The considerations leading to the approximate equation (2.6) may easily be extended to include this differential type of assembly (e. g. ZHOKOVSKII 1960) and lead to the expectation of a distortion coefficient in the region 3 to 4×10^{-7}/bar, with a gradual decrease as the diameter of the upper unit is reduced. Experience at the National Physical Laboratory so far has indicated, however, that this

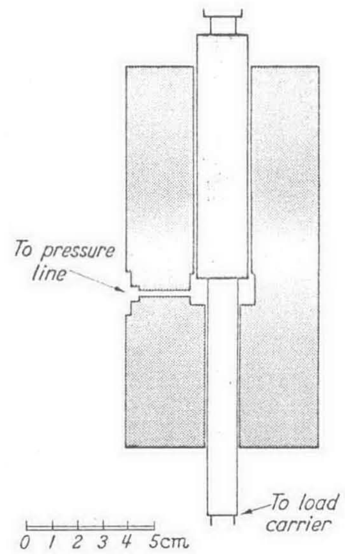

Fig. 5. Diagram of differential piston-cylinder assembly

type of assembly does not exhibit the kind of consistency found in the case of the simple piston-cylinder assemblies. In a group of ten such differential assemblies coefficients ranging from about zero to 11×10^{-7}/bar were found, with no indication of any regular dependence on the constituent piston diameters. This may be due to the fact that in many cases the effective area is the difference between two much larger areas so that the effect of any abnormality on the part of either of the constituent piston-cylinder combinations may be considerably magnified. It could also be associated in part with the difficulty of constructing such assemblies with the two cylinders exactly coaxial. Whatever the explanation, however, it seems that each assembly of this type requires individual calibration and that the assignment of typical values of the distortion coefficient, or reliance on calculated values, would not be satisfactory in this case.

5. The Flow Method

a) *Principle of the method*

The flow method was developed in order to provide an independent check of the changes of effective area of a pressure balance assembly determined by the similarity method, by means which would be independent of the considerations on which the similarity method is based, but which would still depend entirely on the properties of the assembly itself without reference to other standards of pressure.

The principle used is to introduce a deliberate and accurately measurable initial change of effective area — by varying the diameter of one of the components of the assembly — which is made to serve as a reference quantity in terms of which the additional changes of effective area due to pressure may be calculated from measurements of other quantities which vary with the applied pressure.

The procedure used is actually only one of a class of possible methods, of which others will be mentioned below. In the form adopted the rates of flow of the pressure-transmitting fluid through the interspace between the piston and cylinder are measured, at a series of applied pressures, using two alternative pistons having an accurately known difference of daimeter. A simple relation may then be developed connecting the changes of effective area due to distortion with the initial change due to the different piston diameter, and the rates of flow corresponding to the two pistons.

Two other methods of the same general nature, but not depending on flow measurement, were considered and some preliminary experiments carried out. In the first case the quantity measured was the rate of retardation of the rotation speed of the piston and loading weights due to fluid friction in the clearance between piston and cylinder, corresponding to the two piston diameters. It was found, however, that the contribution due to air friction on the rotating load system was an important factor, and rather elaborate measures would have been necessary to eliminate this effect. In the second case the intention was to compare the electrical capacitances of the piston-cylinder assembly corresponding to the two piston diameters. This method, on which so far only very preliminary trials have been made, would very likely repay further exploration, but a knowledge of the pressure dependence of the dielectric constant of the transmitting fluid would be required to complete the reduction of the experimental data.

b) Theory of the flow method

The main problem in the theory of the method is to establish a reasonably simple connection between the measured rates of flow of the pressure transmitting fluid and the corresponding changes of effective area at the same applied pressures.

To introduce the variation of effective area with pressure we adopt the formal expression (2.5) of section 2 b, in which the only term dependent upon h is the integral $\frac{2}{rP}\int_0^P h dp$. The remaining variable term, $P(3\sigma - 1)/E$, is a small part of the total, and it has already been seen that the assumption on which the derivation of this term is based is unlikely to lead to appreciable error.

Denoting by Q the volume velocity of the fluid through any section of the annular gap, and $\eta(x)$ the coefficient of viscosity of the fluid at the axial distance x, it is easily shown that, under conditions of viscous flow,

$$\frac{3Q}{4\pi r} = -\frac{dp}{dx}\frac{h^3}{\eta} \text{ *} \qquad (5.1)$$

* To avoid unnecessarily complicating the notation we ignore variations of the density of the fluid with pressure, as these are very unimportant compared with the variations in the coefficient of viscosity.

and by direct integration, we have

$$\frac{3Ql}{4\pi r} = \int_0^P \frac{h^3}{\eta}\,dp\,. \qquad (5.2)$$

In order to exhibit the direct relation between Q and $\int_0^P hdp$ in a suitable form we may integrate equation (5.1) by a different route, whence we obtain

$$\left(\frac{3Q}{4\pi r}\right)^{\frac{1}{3}} = -\int_0^P hdp \Big/ \int_0^P \left(\eta\frac{dx}{dp}\right)^{\frac{1}{3}} dp\,. \qquad (5.3)$$

This equation shows that the factor relating $Q^{\frac{1}{3}}$ to $\int_0^P hdp$ is a function only of the pressure distribution in the interspace between piston and cylinder, and is not explicitly dependent on h. This suggests the possibility that $\int_0^P \left(\eta\frac{dx}{dp}\right)^{\frac{1}{3}} dp$ may not vary very much for a moderate change in the initial diameter of the piston.

Re-arranging equations (5.2) and (5.3), and writing for brevity

$$\chi = \left(\frac{3Q}{4\pi r}\right)^{\frac{1}{3}} \quad \text{and} \quad I = -\int_0^P \left(\eta\frac{dx}{dp}\right)^{\frac{1}{3}} dp\,,$$

we have

$$\int_0^P hdp = \chi I\,;\; I = l^{\frac{1}{3}}\int_0^P hdp \Big/ \left(\int_0^P \frac{h^3}{\eta}\,dp\right)^{\frac{1}{3}}. \qquad (5.4)$$

The second of these equations provides the basis for the calculation of the integral factor I, connecting the required changes of effective area with the experimentally determined rates of flow.

Before considering further the evaluation of the integral I, it is convenient to convert the formal equations connecting the changes of effective area with the quantities χ and I to a form suitable for application to the experimental data. Proceeding from equation (2.5) and using suffixes 1, 2 where necessary to distinguish the two piston diameters, and denoting by δr the value of $(r_1 - r_2)$ we have

$$A_{P,1} = \pi r_1^2\left[1 + \frac{P}{E}(3\sigma - 1) + \frac{2}{r_1 P}\int_0^P h_1\,dp_1\right],$$

$$A_{P,2} = \pi r_2^2\left[1 + \frac{P}{E}(3\sigma - 1) + \frac{2}{r_2 P}\int_0^P h_2\,dp_2\right]$$

whence, ignoring terms of the second order of small quantities,

$$A_{P,1} + A_{P,2} = 2\pi r_1^2\left[1 + \frac{P}{E}(3\sigma - 1) - \frac{\delta r}{r} + \frac{1}{rP}\left(\int_0^P h_1\,dp_1 + \int_0^P h_2\,dp_2\right)\right],$$

and

$$A_{P,1} - A_{P,2} = 2\pi r_1^2\left[\frac{\delta r}{r} + \frac{1}{rP}\left(\int_0^P h_1\,dp_1 - \int_0^P h_2\,dp_2\right)\right].$$

Transposing, and substituting $\chi_1 I_1$ for $\int_0^P h_1 \, dp_1$ etc. we obtain

$$\frac{\frac{1}{2}(A_{P,1}+A_{P,2}) - \pi r_1^2 \left[1 + \frac{P}{E}(3\sigma - 1) - \frac{\delta r}{r}\right]}{\frac{1}{2}(A_{P,1}-A_{P,2}) - \pi r_1^2 \frac{\delta r}{r}} = -\frac{\left(\frac{\chi_2}{\chi_1}\frac{I_2}{I_1}+1\right)}{\left(\frac{\chi_2}{\chi_1}\frac{I_2}{I_1}-1\right)}. \tag{5.5}$$

The right hand side of this equation is completely determined by χ_2/χ_1, the ratio of the cube roots of the measured rates of flow, and the ratio I_2/I_1. The quantity $(A_{P,1} - A_{P,2})$ occurring in the denominator of the left hand side may be determined by direct balancing of the two forms of the assembly against any third reference assembly, and the quantity $\delta r/r$ is established by diametral measurements on the two pistons. The small quantity $P(3\sigma - 1)/E$ is known with sufficient accuracy from the elastic constants of the material. Subject, therefore, to further examination of the factor I_2/I_1 equation (5.5) enables the quantity $(A_{P,1} + A_{P,2})/2$, i. e. the mean of the effective areas of the two forms of the assembly, to be determined, as a function of the applied pressure, from the experimental observations.

It is evident that the term $\pi r_1^2 \cdot \delta r/r$ occurring in the denominator of the left hand side should be identical with $(A_{0,1} - A_{0,2})$ and this may be checked directly from the experimental data. If, as may be the case, the difference $(A_{P,1} - A_{P,2})$ is independent of pressure, the denominator of the left hand side may be written more simply as $-(A_{0,1} - A_{0,2})/2$, but it cannot of course be assumed a priori that this condition will hold.

c) *Treatment of the integral 'I'*

In order to estimate the value of I some simplifying assumptions must be introduced if the theory is not to become unjustifiably complicated. From the second of equations (5.4) it is clear that we can calculate I if we can express h and η as functions only of p. As regards h, the justification for assuming that the part of $h(x)$ arising from distortion due to the pressure in the interspace between the piston and cylinder may be taken as proportional to the pressure $p(x)$ at the same position has already been discussed. Bearing in mind that we are not really interested in the absolute values of I_1 and I_2, but only in their ratio, this assumption is not likely to lead us far astray. As before, there is an additional component of h arising from the longitudinal thrust on the piston, which will be proportional to the total applied pressure, P. We therefore write

$$h = H + \nu P + \mu p \tag{5.6}$$

where μ and ν are constants.

The coefficient of viscosity at constant temperature is certainly determined uniquely by the pressure and there is considerable evidence available from published measurements that the dependence may be represented reasonably closely by an exponential function, in other words that we may write

$$\eta = \eta_0 \, e^{\alpha p} \tag{5.7}$$

where α is a constant and η_0 is the value at zero (or atmospheric) pressure. This relation has been found to hold with fair accuracy for most oils of types likely to be used in conjunction with pressure balances, although it appears that there may be more pronounced departures in the case of some silicone fluids (BRIDGMAN 1952; Amer. Soc. Mech. Engrs. 1953; ZOLOTYKH 1960).

The evaluation of I in terms of the constants in equations (5.6) and (5.7) is now straightforward and, writing for brevity

$$c = (H + \nu P)\,\alpha/\mu \ ,$$

we obtain

$$I = (l\eta_0)^{\frac{1}{3}} P^{\frac{3}{2}} I' \ ,$$

where

$$I' = (\alpha P)^{\frac{1}{3}}\left(c + \frac{\alpha P}{2}\right) \begin{Bmatrix} c^3 + 3c^2 + 6c + 6 \\ -e^{-\alpha P}[(c+\alpha P)^3 + 3(c+\alpha P)^2 + 6(c+\alpha P) + 6] \end{Bmatrix}^{-\frac{1}{3}} \tag{5.8}$$

This quantity may conveniently be represented as a family of graphs showing its dependence on $(H + \nu P)/\mu P$ for a suitable range of values of αP.

In order to apply equation (5.8) the values of $(H + \nu P)/\mu P$ and αP corresponding to the experimental points are required. Denoting by ϱ_P the ratio $\chi_2 I_2/\chi_1 I_1$ at a given applied pressure P, and by ϱ_0 the extrapolated value of ϱ_P corresponding to zero applied pressure, and using equation (5.6), we find that

$$\varrho_P = \frac{H_2 + \nu P + \mu P/2}{H_1 + \nu P + \mu P/2} \ ,$$

whence, after some reduction, we obtain the equations

$$\frac{H_1 + \nu P}{\mu P} = \frac{\varrho_0 - 1}{\varrho_0 - \varrho_P}\left(\frac{\nu}{\mu} + \frac{1}{2}\right) - \frac{1}{2} \tag{5.9}$$

and

$$\frac{H_2 + \nu P}{\mu P} = \varrho_P \frac{\varrho_0 - 1}{\varrho_0 - \varrho_P}\left(\frac{\nu}{\mu} + \frac{1}{2}\right) - \frac{1}{2} \ . \tag{5.10}$$

We do not need to know the values of μ and ν, but an approximate figure for ν/μ is required. From the elementary theory leading to equation (2.6) we easily find

$$\nu/\mu = -\sigma/2 \text{ (approx.)}$$

whence we obtain, with sufficient accuracy, $\nu/\mu = -0.15$. Initially, of course, we cannot actually use the ratio $\chi_2 I_2/\chi_1 I_1$ since I_2/I_1 has not yet been determined. In practice, therefore, we commence by using simply the experimental ratio χ_2/χ_1 to obtain a first approximation to the correction factor, and then if necessary proceed to a second approximation.

To derive the appropriate value of αP we again ignore initially the distinction between $I_1(P)$ and $I_2(P)$, and denoting the quantity $(\chi_2 - \chi_1) P^{-\frac{1}{3}}$ at the applied pressure P by $\Delta\chi(P)$, we obtain from equations (5.4) and (5.6)

$$\Delta\chi(P) = P^{\frac{2}{3}}(H_2 - H_1)/I(P) = (H_2 - H_1)(l\eta_0)^{-\frac{1}{3}}/I'(P)$$

whence, dividing the experimental value of $\Delta\chi(P)$ into the extrapolated value corresponding to zero applied pressure, we have

$$\Delta\chi(O)/\Delta\chi(P) = I'(P)/I'(O) \ .$$

Making use of the values of $(H + \nu P)/\mu P$ derived above we are now able to determine the value of αP which best fits the experimental data by simple inspection of graphs or tables of the function I'. In this case, again, a second approximation may be derived if necessary.

Having carried out the above procedures we are now in a position to determine the values of I_1 and I_2, and consequently the ratio I_2/I_1, corresponding to the actual experimental points, and then to calculate, using equation (5.5) of the previous section, the changes in the effective areas of the assemblies as a function of the applied pressure.

d) *Experimental method*

The experimental procedures used in that part of the flow method involving the direct comparison of effective areas by balancing need no further consideration as they are exactly the same as those previously described in section 3 f. In the measurement of the rate of flow of the pressure balance fluid use is made of a very simple device. With the balance operating on an otherwise leak-proof system the change in volume of the contained fluid due to flow through the interspace between the piston and cylinder is exactly compensated by the gradual descent of the piston, and the rate of fall of the latter is thus directly proportional to the rate of flow. All that is necessary therefore is to time the descent of the piston over a constant short distance, the measured time being inversely proportional to the rate of flow. In practice this was carried out by using an optical magnification system and measuring the time of descent over a distance of the order 1 mm by stopwatch, but if the method were to be used at all extensively a more sophisticated procedure could obviously be substituted, using, for example, a photoelectric recording arrangement.

The work has been carried out using two piston-cylinder assemblies, of nominal effective areas 0.05 and 0.02 in², covering respectively pressure ranges up to about 600 bars and 1500 bars. The transmitting fluid used was in each case a mixture of two mineral oils, known commercially as Diala and Talpa respectively.

Since the coefficient of viscosity is markedly dependent on temperature precautions had to be taken to ensure that the temperature of the piston-cylinder assembly remained as constant as possible during a series of measurements. As a fully temperature-controlled room was not available it was necessary to determine a temperature coefficient in order that each series of readings could be converted to a common temperature, which was taken to be 20 °C. In general the apparatus and air temperatures were held to within a few tenths of a degree during any one series. In order to avoid extraneous friction, all the measurements were made with the piston and associated load in free rotation, the speed chosen being in the range 30 to 40 rev/min. In any group of measurements at a given pressure readings were taken alternately for the two directions of rotation, and the mean taken, to ensure that any possible effects due to small helical errors on the piston surface were eliminated.

The changes of effective areas with pressure were also measured, using the same pressure transmitting fluid in each case, by the similarity method. The results of the measurements, and the comparison of the two methods, are discussed in the next section.

6. Results of the Flow Method

a) *Experimental parameters and correction terms*

The various parameters and correction terms required in the derivation of the changes of effective area as a function of pressure are given in Tab. 3 for the two assemblies concerned, together with the distortion coefficients determined by the flow and similarity methods.

Table 3. *Parameters and correction terms*

Nominal area of assembly	Mean difference of piston diameters	$\frac{A_{0\,(1)} - A_{0\,(2)}}{A_0} = \frac{\delta r}{r}$ calculated from difference of piston diameters (parts in 10^5)	$\frac{A_{P\,(1)} - A_{P\,(2)}}{A_0}$ experimental value (parts in 10^5)	ϱ_0 (extrap)	Typical values of correction term I_2/I_1: Pressure (bar)	I_2/I_1	Estimated value of α (bar^{-1})	Distortion coefficient (bar^{-1}): Flow method	Similarity method
0.05 in² (0.322 cm² approx.)	5.45×10^{-5} in ($13.8_4 \times 10^{-5}$ cm)	21.6	21.4 (independent of pressure)	1.48	0 140 280 560	1.000 1.002 1.005_5 1.004	3.2×10^{-3}	$4.2_5 \times 10^{-7}$	$4.3_8 \times 10^{-7}$
0.02 in² (0.129 cm² approx.)	2.75×10^{-5} in ($6.9_8 \times 10^{-5}$ cm)	17.5	17.5 (P = 0) decreasing smoothly to 15·0 (P=1500)	1.32	0 250 500 1000 1500	1.000 1.001_5 1.003 1.002 0.999	2.3×10^{-3}	$4.0_7 \times 10^{-7}$	$3.9_6 \times 10^{-7}$

A good check of the internal consistency of the different measurements is provided by a comparison of the figures in the third and fourth columns of the Table from which it will be seen that the changes of effective area at zero pressure, calculated from the measured piston diameters, are in very close agreement with the changes determined experimentally by direct balancing. The correction terms I_2/I_1, of which typical values are given, nowhere differ from unity by as much as 1% in the present range of experiments, but owing to the form of the right hand side of equation (5.5) they are just sufficiently significant to warrant taking them into account. If the flow method were to be extended to higher pressure ranges it is likely that larger corrections would be involved, and these might eventually limit the pressure range attainable with reliability.

It will be seen that somewhat different values of the coefficient α were found in the two cases. This probably arose from the fact that the transmitting fluids, although nominally similar mixtures, were not actually identical, as was apparent from differences of viscosity.

b) Comparison with the similarity method

The most important aspect of the flow method as so far applied is its value as confirmation of the results obtained by the similarity method. This comparison is shown, for the two cases considered, in Tab. 3 and in Figs. 6 and 7. In each case the same piston-cylinder assembly, and the same transmitting fluid, were used in the determinations by the two methods. Although

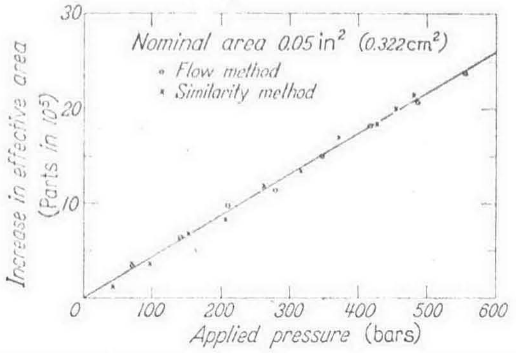

Fig. 6. Comparison of distortion factors determined by the similarity and flow methods. Assembly of type a with mineral oil mixture A

the agreement is very close in both cases, the confirmation is especially important in the case illustrated in Fig. 7 where the pressure, and the corresponding distortion factors, cover the widest range. The flow method confirms not only the value of the distortion coefficient, with agreement to the order 2 or 3% in

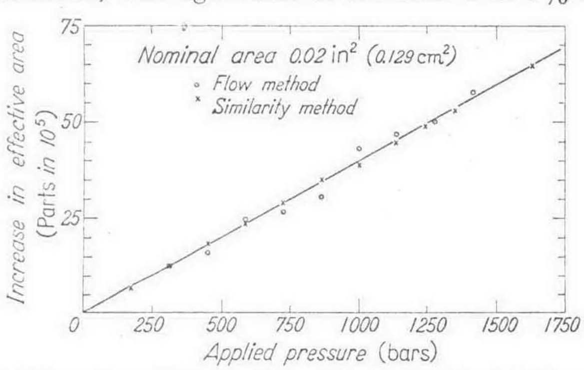

Fig. 7. Comparison of distortion factors determined by the similarity and flow methods. Assembly of type a with mineral oil mixture B

both cases, but also the fact that the distortion term is accurately representable as a linear function of the applied pressure. The flow method shows a slightly increased dispersion of the experimental points, which is believed to be due to residual uncertainties in the temperature of the assembly. It has been noted that the correction terms I_2/I_1 do not differ greatly from unity, but it is found that the inclusion of this correction factor gives a small but definite improvement in the agreement with the similarity method.

The results of these comparisons therefore support the estimate of accuracy of the measured distortion coefficients already arrived at as a result of the discussion in Section 4 b, viz. that these coefficients are determined to about 2%. An error of this magnitude in the distortion coefficient would imply an error of about 1 part in 10^5 in the effective area of the assembly at 1000 bars compared with its value at zero pressure, the error increasing in proportion to the applied pressure.

The very close agreement between the flow method and the similarity method up to the region of 1500 bars suggests that the flow method, or one of the other methods based on the same general principle, may have useful applications in the further extension of this work to higher pressures.

Acknowledgements. The authors are glad to acknowledge the ready assistance provided by Mr. H. L. COX, Mr. A. F. C. BROWN and Mr. M. F. MARKHAM of the Materials Section of the Basic Physics Division of the Laboratory in the measurement of elastic constants and on many questions involving elastic theory. The special pistons and cylinders needed in the many measurements carried out were constructed in the Laboratory's precision workshops, and special acknowledgement is due to Mr. E. H. PINN and the late Mr. J. R. WHITE both on account of their advice on various problems and the very careful and precise constructional work involved. The assistance of several members of the Engineering Metrology Section of the Standards Division of the Laboratory in making the highly accurate diametral measurements on pistons and cylinders has also been an important contribution to the work.

The work described in this paper forms part of the research programme of the National Physical Laboratory, and is published by permission of the Director.

References: Amer. Soc. Mech. Engrs., Research Publication, Pressure-Viscosity Rep. Vols. I and II, New York 1953. – BARTON, M. V.: J. Appl. Mechanics, Trans. Am. Soc. Mech. Engrs. 8, A-97 (1941). – BEATTIE, J. A., and W. L. EDEL: Ann. Phys., **11**, 633 (1931). – BENNETT, C. O., and B. VODAR: Rep. of High-Pressure Measurement Symposium, Am. Soc. Mech. Engrs., New York, November 1962, 365, (Butterworths, Wash. 1963). – BETT, K. E., P. F. HAYES, and D. M. NEWITT: Phil. Trans. Roy. Soc. (A) **247**, 59 (1954). – BETT, K. E., and D. M. NEWITT: Rep. of Symposium on The Physics and Chemistry of High Pressures, Lond., June 1962, 99, Soc. of Chem. Ind., Lond. (1963). – BRIDGMAN, P. W.: The Physics of High Pressure. London: Bell 1952. – BROWN, A. F. C., A. G. COLE, and M. F. MARKHAM: Nature (Lond.) **180**, 1254 (1957). – CHREE, C.: Trans. Cambridge Phil. Soc. **XIV**, 250 (1889). – Phil. Mag. (6) **2**, 532 (1901). – CROMMELIN, C. A., and E. I. SMID: Comm. Phys. Labty. Univ. Leyden, No. 146C (1915). – DADSON, R. S.: Nature (Lond.) **176**, 188 (1955). – Rep. of Conf. on Thermodynamic and Transport Properties of Fluids, Lond., Inst. Mech. Engrs., 37 (1958). – EBERT, H.: Physik. Z. **36**, 385 (1935). – Z. angew. Phys. **1**, 331 (1949). – Arch. techn. Messen V, 1340-1. (1951). – ELLIOTT, K. W. T., D. C. WILSON, F. C. P. MASON, and P. H. BIGG: J. sci. Instruments **37**, 162 (1960). – FILON, L. N. G.: Phil. Trans. Roy. Soc. (A) **198**, 147 (1902). – HOLBORN, L., u. H. SCHULZE: Ann. Physik **47**, 1089 (1915). – JOHNSON, D. P., J. L. CROSS, J. D. HILL, and H. A. BOWMAN: Ind. Engng. Chem. **49**, 2046 (1957). – JOHNSON, D. P., and D. H. NEWHALL: Trans. Am. Soc. Mech. Engrs. **75**, 3, 301 (1953). – KEYES, F. G., and J. DEWEY: J. opt. Soc. Amer. **14**, 491 (1927). – LOVE, A. E. H.: The Mathematical Theory of Elasticity, 4th Edn. Cambridge 1952. – MARKHAM, M. F.: Brit. J. appl. Physics Suppl. No. **6**, 56 (1957). – MEYERS, C. H., and R. S. JESSUP: J. Res. nat. Bur. Standards **6**, 1061 (1931). – MICHELS, A.: Ann. Physik **72**, 285 (1923); **73**, 577 (1924). – Proc. Acad. Sci. (Amsterdam) **35**, 994 (1932). – National Physical Laboratory, Ann. Rep. Series, 86 (1919). – RANKIN, A. W.: J. Appl. Mechanics, Trans. Am. Soc. Mech. Engrs. **2**, A-77 (1944). – ROEBUCK, J. R., and W. CRAM: Rev. Sci. Instr. **8**, 215 (1937). – ROEBUCK, J. R., and H. W. IBSER: Rev. Sci. Instr. **25**, 46 (1954). – SAMOILOV, V. N.: Trudy Inst. Komiteta **46** (106), 43 (1960). – SEARS, J. E. (jun.), and J. S. CLARK: Proc. Roy. Soc. (A) **139**, 130 (1933). – TOYOSAWA, Y.: Bull. Nat. Res. Lab. Metrol. (Tokyo) No. 6 abstracts of reports of Cent. Insp. Inst. of Wts & Meas. Nos. 19 (Vol. 8 1959) and 23 (Vol. 9 1960), (1963). – Bull. Nat. Res. Lab. Metrol. (Tokyo) No. 8 (1964). – TRANTER, C. J.: Quart. Appl. Math. **4**, 298 (1946). – TRANTER, C. J., and J. W. CRAGGS: Phil. Mag. (7), **38**, 214 (1947). – ZHOKHOVSKII, M. K.: Trudy Inst. Komiteta **46**, (106), 5, 30 (1960). – ZOLOTYKH, E. V.: Trudy Inst. Komiteta **46**, (106), 81 (1960).

R. S. DADSON, R. G. P. GREIG, ANGELA HORNER
Standards Division, National Physical Laboratory,
Teddington/Middlesex